Gravitational Systems Simulator

ASTR311

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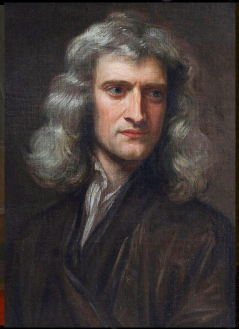
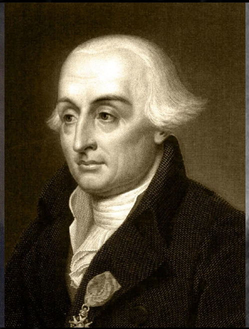
From Kepler to Chaos

Kepler
1609



Newton
1687



Lagrange
1772

Poincaré
1890s

Kepler figured out how planets orbit, Newton figured out why they orbit. Euler and Lagrange derive special cases for orbiting bodies, and Poinecaré showed that the problem has no closed form solution.

1609 **Kepler**

Planets move in ellipses, Kepler's laws

1687 **Newton**

Inverse square gravity solves the 2-body problem

1700s **Euler + Lagrange**

Find special solutions to the 3-body problem

1890s **Poincaré**

Shows the general case is not integrable

Today **Still Open**

No general close-form solutions exist, only approximation and special cases

Refs: [1.1], [1.2], [1.3], [1.4]

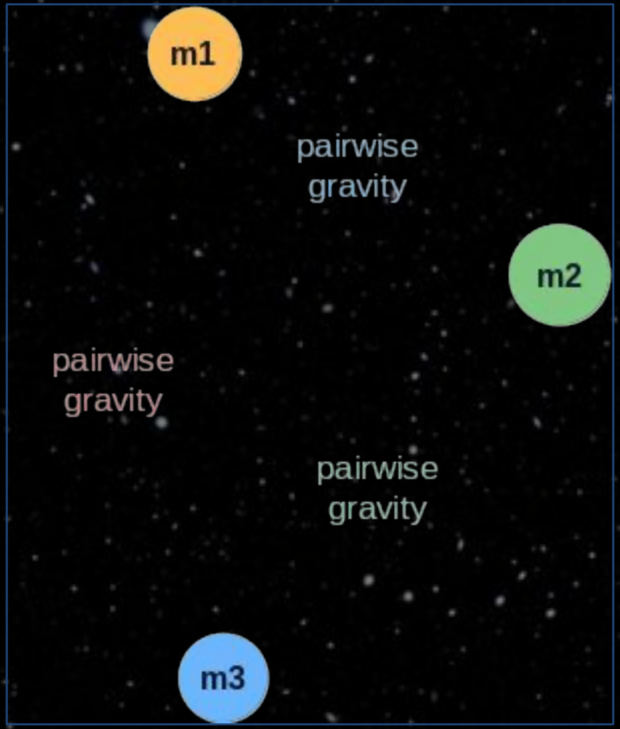
What is the three-body problem?

- The two-body problem, with just two particles is a solved problem.
- “Three particles move in space under their mutual gravitational attraction; given their initial conditions, determine their subsequent motion.” (Barrows-Green 1933)
- Each body feels the vector sum of the gravitational pulls from the other two bodies
- Newtonian equations for three bodies, where, $\mathbf{r}_i = (x_i, y_i, z_i)$ and with masses m_i :
- No exact solution for these equations but can be approximated using various methods

$$\ddot{\mathbf{r}}_1 = -Gm_2\frac{(\mathbf{r}_1-\mathbf{r}_2)}{|\mathbf{r}_1-\mathbf{r}_2|^3} - Gm_3\frac{(\mathbf{r}_1-\mathbf{r}_3)}{|\mathbf{r}_1-\mathbf{r}_3|^3}$$

$$\ddot{\mathbf{r}}_2 = -Gm_3\frac{(\mathbf{r}_2-\mathbf{r}_3)}{|\mathbf{r}_2-\mathbf{r}_3|^3} - Gm_1\frac{(\mathbf{r}_2-\mathbf{r}_1)}{|\mathbf{r}_2-\mathbf{r}_1|^3}$$

$$\ddot{\mathbf{r}}_3 = -Gm_1\frac{(\mathbf{r}_3-\mathbf{r}_1)}{|\mathbf{r}_3-\mathbf{r}_1|^3} - Gm_2\frac{(\mathbf{r}_3-\mathbf{r}_2)}{|\mathbf{r}_3-\mathbf{r}_2|^3}$$



Refs: [1.2], [1.3]

Chaos, Stability, and Special Cases

Chaos

- Nearby initial states can separate exponentially in chaotic regions (i.e. points of maximum gravity)
- Deterministic does not necessarily lead to predictability
- For strong non-hierarchical encounters, a common outcomes is the ejection of a third body leaving a binary

Special Solutions

- Euler and Lagrange families because they are symmetric in their motion
- E.g. Equal-mass figure-8 orbit and periodic orbits
- Periodic or resonant motion exists but is rare compared to the generic case

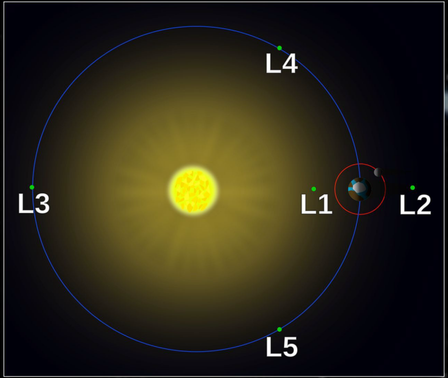
Real Systems

- Long lived systems owe their survival to hierarchy, resonance, or both
- The solar system is not perfectly regular!
- Long term integrations shows chaotic behaviours (just on very long time scales)
- Resonance and chaos are not mutually exclusive, see pluto's moons as an example

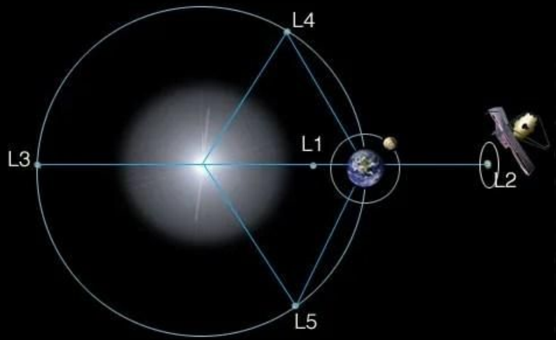
Refs: [1.3], [1.5], [1.6], [1.7], [1.14]

Restricted Three Body Problem

- A simplification of the three-body problem
- Assume one body is so small that it does not affect the motion of the other two bodies
- This produces a problem that is much easier to analyze than the general case
- In a rotating frame, the system reveals the five stable lagrange points

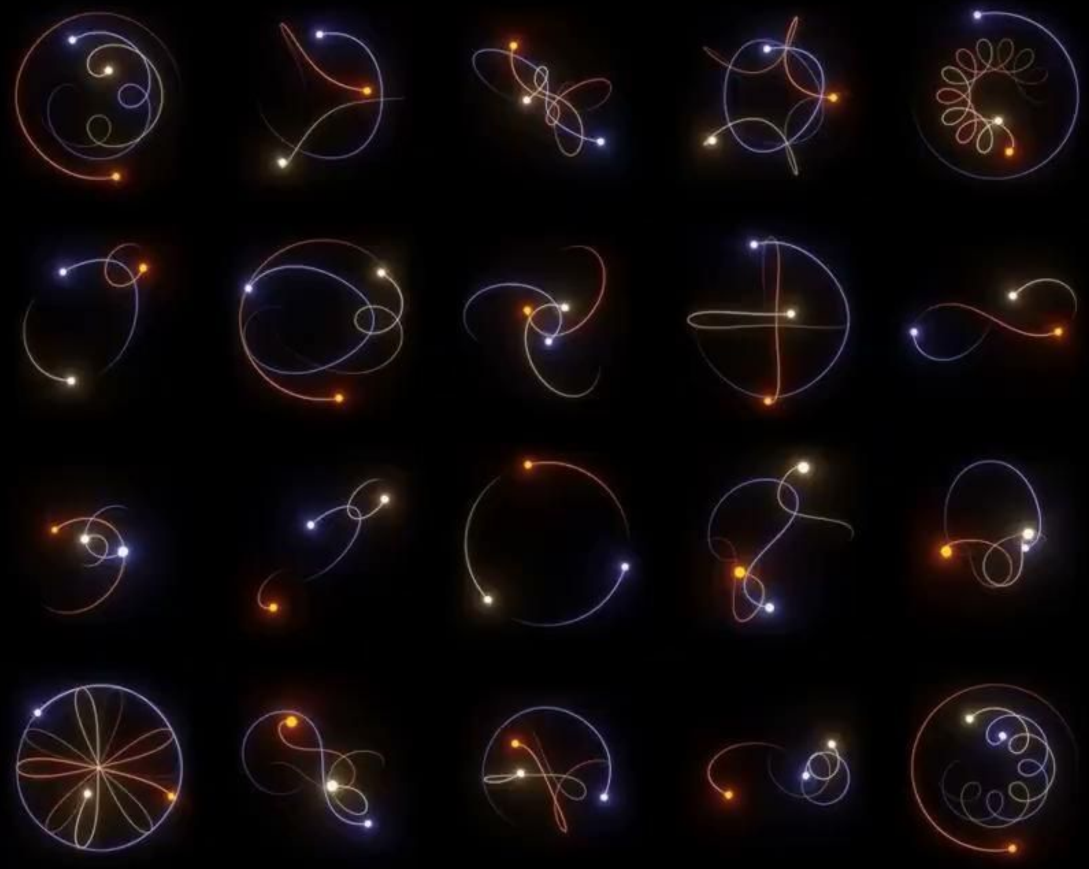


https://commons.wikimedia.org/wiki/File:Lagrange_poi



https://science.nasa.gov/mission/webb/orbit/

Refs: [3], [11]



https://en.wikipedia.org/wiki/File:5_4_800_36_downscaled.gif

How did I approximate the simulation of the three-body problem?

To use symplectic mapping or just approximate position however.

Model: classical mutual Newtonian gravity in 2D (planar N-body), each body feels the vector sum of forces from every other body

- I.e. inverse square law with Plummer softening

Force is evaluated as a direct summation of the gravitational acceleration of each body: this is ok for small N

Time integration is done with Velocity Verlet i.e. Stormer-Verlet

Output of the calculations is a precomputed trajectory for each body that is sent to the frontend/client and is rendered from there using JavaScript

Calculation are performed using python and NumPy

1. Newtonian and second order ordinary differential equations

$$\ddot{\mathbf{r}}_i = G\sum_{j\neq i} m_j \, \mathbf{r}_{ji}/r_{ji}^3$$

2. First order state vector (i.e. R^18 3x3x3, 3 body, 3 position, 3 velocities)

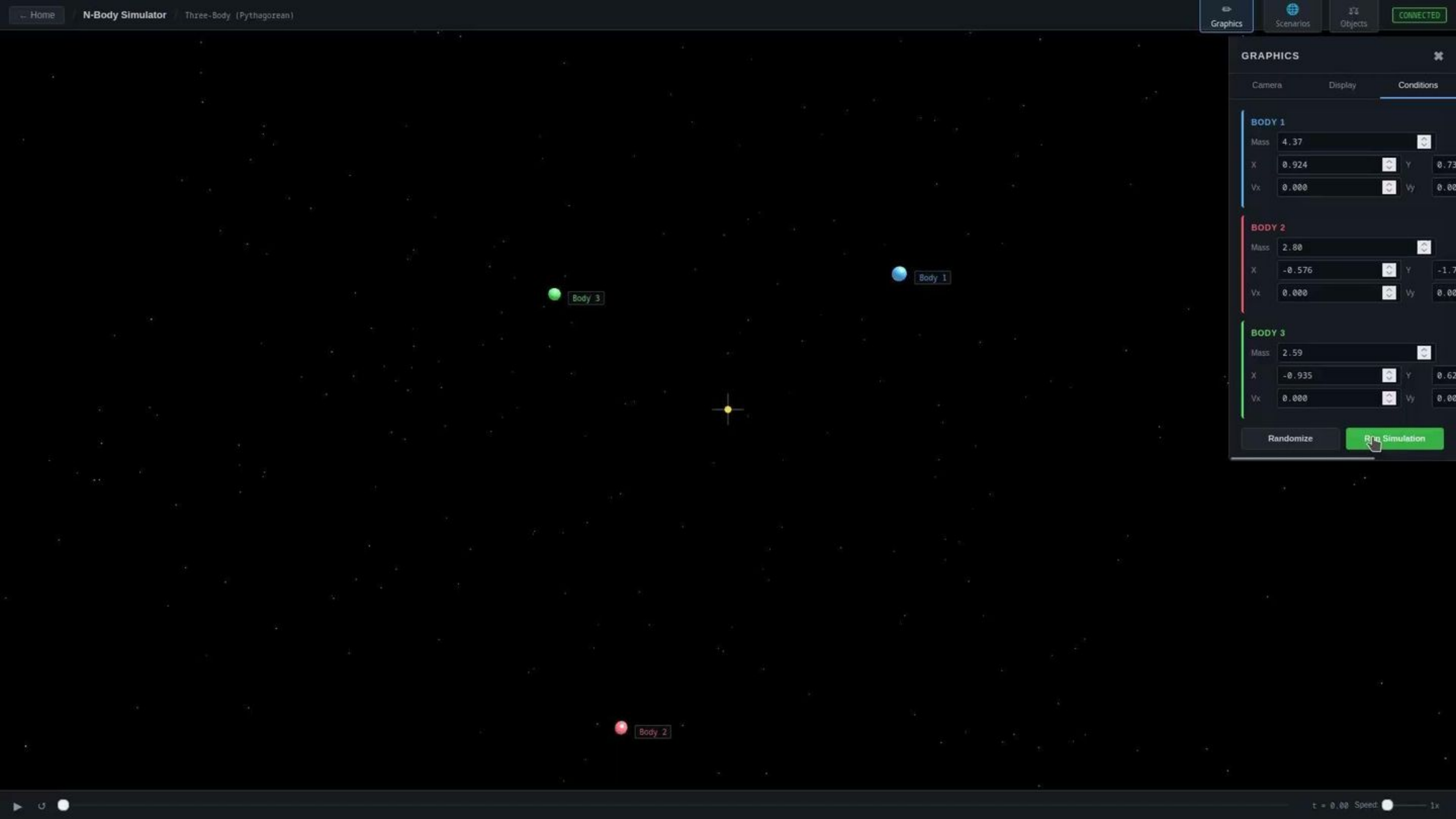
$$\dot{\mathbf{x}} = f(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^{18}$$

3. Hamiltonian mechanics

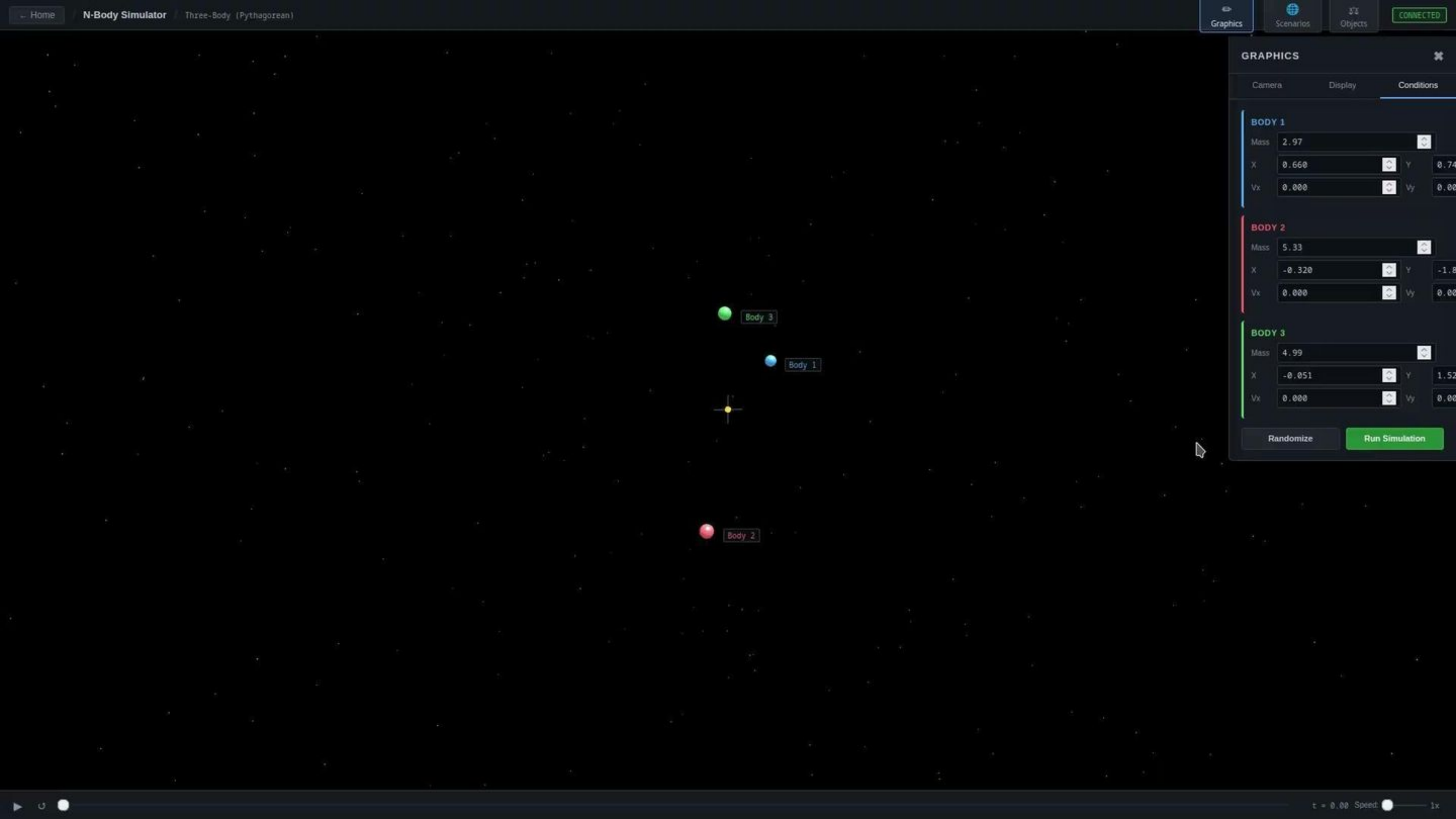
$$H = \sum_i p_i^2/(2m_i) - \sum_{i<j} G m_i m_j / r_{ij}$$

4. Other numerical methods
   a. Runge-Kutta: generic ODE solver
   b. Sympletic methods: long term orbital structure
   c. Regularization / adaptives: close encounters causing extreme changes in position (i.e. ejection)

Refs: [3], [8], [9], [10]



← Home
N-Body Simulator
Three-Body (Pythagorean)
Graphics
Scenarios
Objects
CONNECTED
GRAPHICS
Camera
Display
Conditions
BODY 1
Mass 4.37
X 0.924
Y 0.73
Vx 0.000
Vy 0.00
BODY 2
Mass 2.80
X -0.576
Y -1.7
Vx 0.000
Vy 0.00
BODY 3
Mass 2.59
X -0.935
Y 0.62
Vx 0.000
Vy 0.00
Randomize
Run Simulation
Body 1
Body 2
Body 3
t = 0.00
Speed
1x


← Home
N-Body Simulator
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Graphics
Scenarios
Objects
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Camera
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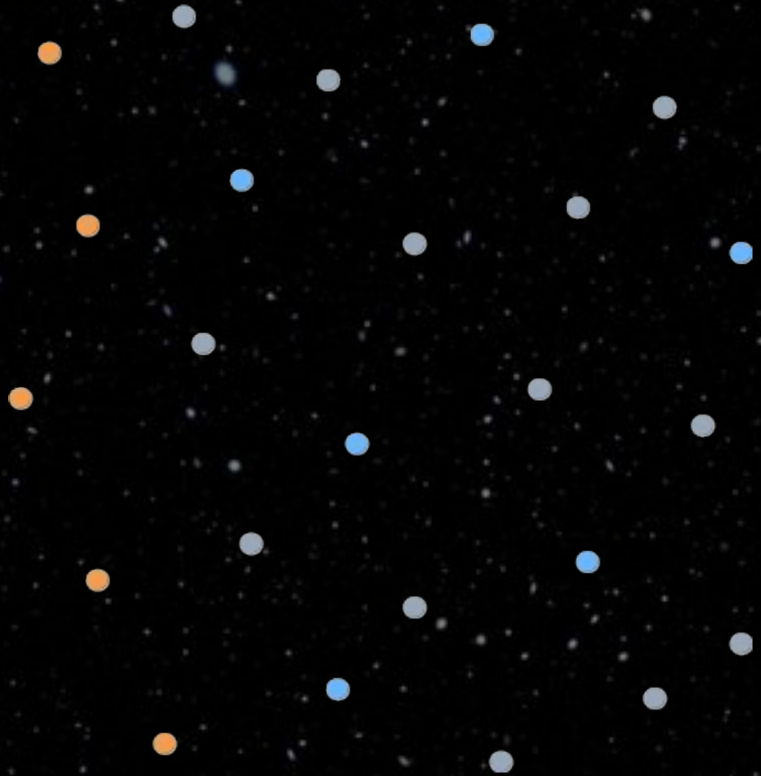
Conditions
BODY 1
Mass 2.97
X 0.660
Y 0.74
Vx 0.000
Vy 0.00
BODY 2
Mass 5.33
X -0.320
Y -1.8
Vx 0.000
Vy 0.00
BODY 3
Mass 4.99
X -0.051
Y 1.52
Vx 0.000
Vy 0.00
Randomize
Run Simulation
Body 3
Body 1
Body 2
t = 0.00
Speed
1x

From 3 Bodies to N Bodies

- The three body problem is a special case of the more general N body problem
- Each particle or body still feels the sum of gravitational pull from all other bodies
- Since the gravitational attraction is pair wise adding extra bodies is computational taxing
- Same rules but more interactions by a factor of N^2



Simulating the Big Bang

- **Millennium Simulation (Springel et al., 2005):**
  - 10 billion particles on supercomputers
  - reproducing galaxies, filaments, and voids from early-universe conditions

- **Goal:** start from nearly uniform matter and watch gravity build large-scale structure

- Even with months of compute time, the outcome depends on **initial conditions** and **expansion rate**

- **Ours: 1000s** of particles on personal server, same concept, different scale

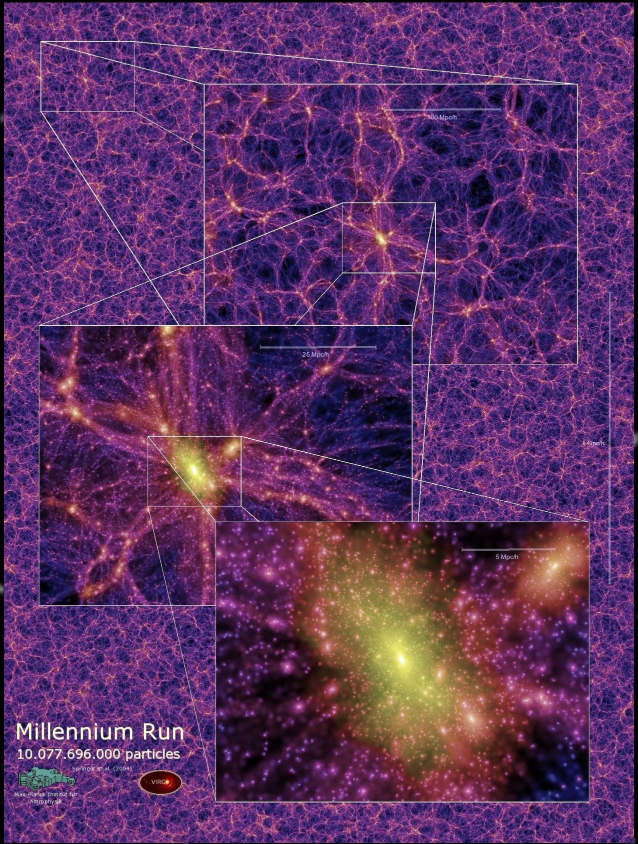


Figure 2.1 The Millennium Simulation Project
Retrieved from:
https://wwwmpa.mpa-garching.mpg.de/galform/virgo/millennium/

Refs: [1.13]

Attempt 1: Newtonian Gravity Simulation

- **Particles start in a dense cloud, expand outward under mutual gravity**
  - **Leapfrog + softening to prevent weird interactions at close range**

- **What emerged: gravitational clumping, slingshot ejections, a bound rotating core**

- **These are real phenomena:**
  - galaxy formations,
  - dynamical ejection

- **The problem:** model has a center, an edge, particles moving through space

- The **Big Bang** is **not** an explosion; **space itself expanded** (no center, no edge)

Refs: [2.01], [2.02]

Attempt 1: The Physics Engine

- **Softened Newtonian gravity**

$$\mathbf{a}i = G\sum j \neq i m_j \frac{\mathbf{r}_j - \mathbf{r}_i}{(|\mathbf{r}_j - \mathbf{r}_i|^2 + \varepsilon^2)^{3/2}}$$

- **ε** is softening length; prevents singularities when particles get too close

- Same structure as Ethan's 3-body equations, but summed over all N pairs
- Complexity: **O(N²)** pairwise interactions

$$\underbrace{N^2}_{\text{pairs}} \times \underbrace{S}_{\text{steps}} = 5000^2 \times 20{,}000 = 5 \times 10^{11} \text{ force evaluations } \approx 50 \text{ hours}$$

500,000,000,000 force evaluations

Refs: [1.9]

Attempt 1: The Demo

gravity.chrislawrence.ca

Figure 2.2 Screenshot of the Newtonian N-body simulation web viewer showing gravitational clumping and ejection. Original work by Chris Lawrence. Available at gravity.chrislawrence.ca.

Refs: []

Attempt 2: Spacetime Expansion

- Space itself stretches via a scale factor **a(t)** that grows over time
- Particles sit at fixed "comoving coordinates" carried apart by expanding space

- **Hubble's Law**
    - Recession velocity = **H(t)** × distance

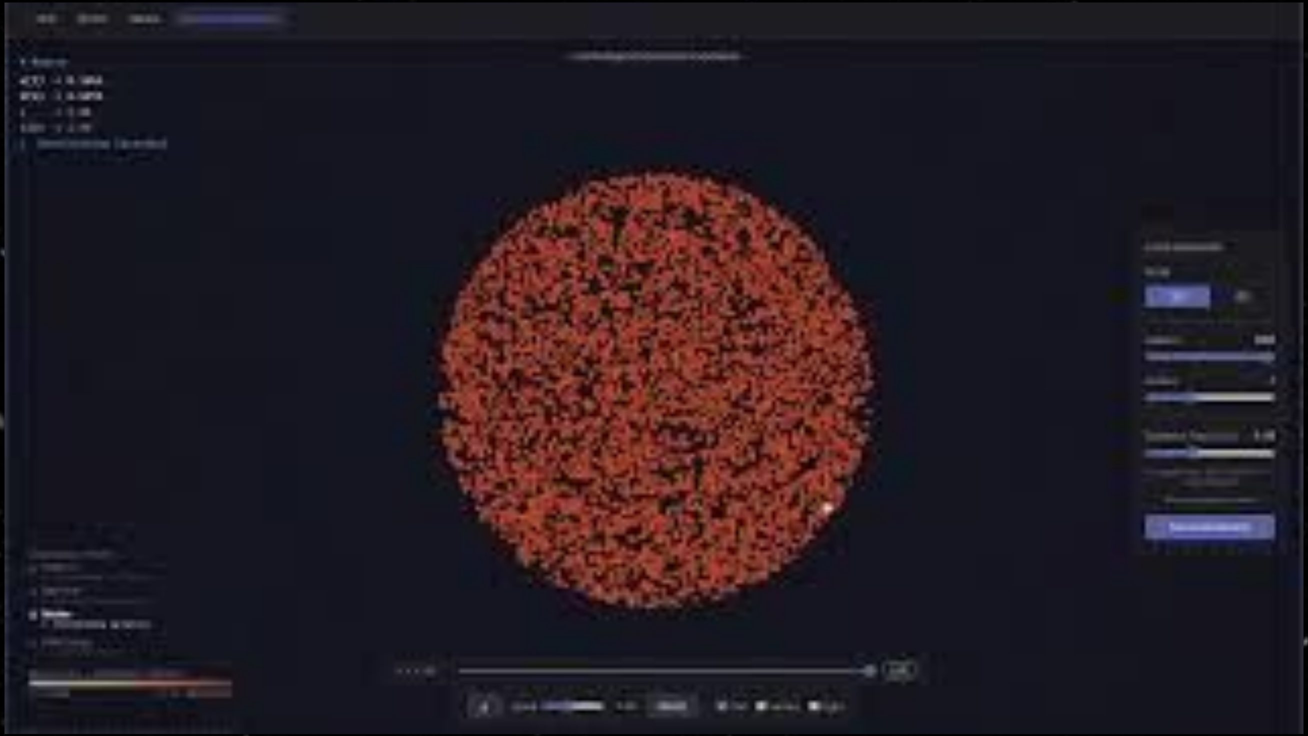
- **No center, no edge, no preferred vantage point**
    - **every observer sees every other galaxy receding**

- **Farther apart = more expanding space in between = faster recession**

- **The expansion rate is the initial condition, it sets everything**

Refs: []

Attempt 2: The Demo



astr311.chrislawrence.ca

Figure 2.3 Screenshot of the spacetime expansion simulation showing Friedmann-driven expansion. Original work by Chris Lawrence. Available at astr311.chrislawrence.ca.

Attempt 2: The Friedmann Equation

The key equation (drives everything in the simulation):

$$H^2(a) = H_0^2 \left[\frac{\Omega_r}{a^4} + \frac{\Omega_m}{a^3} + \Omega_\Lambda \right]$$

H(t) = expansion rate, a(t) = scale factor, each term dominates at a different era

Four cosmological epochs; what dominates the right-hand side changes everything:

Era	Dominant term	Expansion	Why
Inflation	Vacuum energy (Λ)	↑↑ Exponential	Constant H → runaway growth
Radiation	$\Omega r / a^4$	↓ Decelerates	Photon pressure, H falls fast
Matter	$\Omega m / a^3$	↓ Slows	Gravity wins, structure forms
Dark Energy	ΩΛ	↑ Accelerates	Λ constant, gravity loses

Refs: [2.03]

What The Models Show, and What They Can't

- **Explosion model:** captures real local gravitational dynamics, but has a center and particles moving through space

- **Expansion model:** captures the right conceptual framework, but can't reproduce filament structure at this scale

- **Real cosmological simulations** need **~10^{10}** particles and months of compute

- **What both models demonstrate:**
  the expansion rate completely changes the outcome

Initial Conditions Determine Everything

Refs: []

Einstein's Theory of Gravity

Einstein's General Relativity: Gravity as Geometry

- The Newtonian picture (1687): Gravity described as an instantaneous force between two masses.
- **The crack in the theory:**
  - Mercury's orbit precesses 43 arcseconds/century MORE than predicted.
  - No planet, hidden mass, or mathematical trick could fix it.
  - For 200 years this was an unresolved wound in Newtonian mechanics.
- **The Einstein discovery (1905, 1915)**
  - Einstein abandoned the idea of gravity as a force altogether.
  - Mass and energy tell spacetime how to curve.
  - Curved spacetime tells matter how to move.
  - Gravity IS the curvature, not a force acting through space.

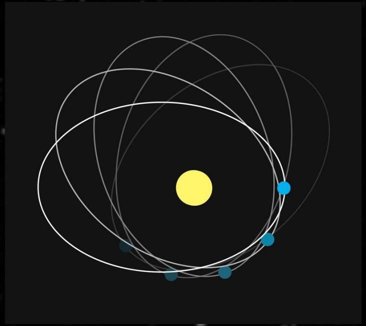


Figure 3.1: Perihelion precession of Mercury [Digital illustration]. Wikimedia Commons. Retrieved from https://en.wikipedia.org/wiki/Tests_of_general_relativity

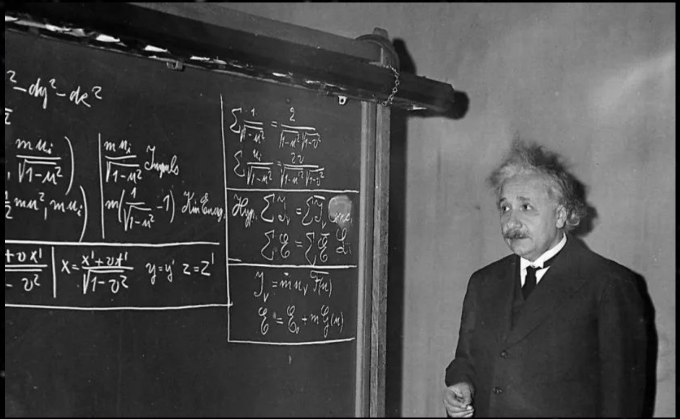
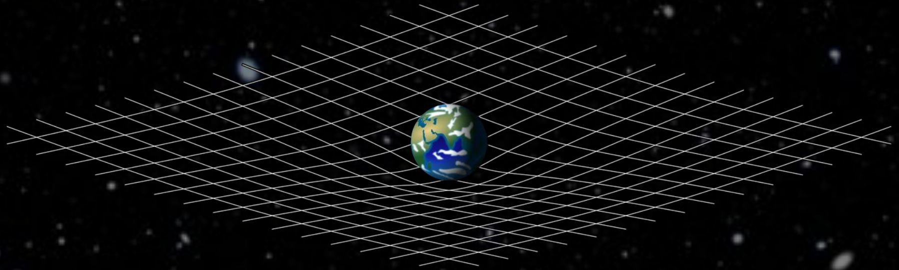


Figure 3.2: Einstein, A. (1934). *Albert Einstein at Carnegie Mellon, Pittsburgh* [Photograph]. Credit: Dwight Vincent & David Topper.

Ripples in Spacetime: Einstein's 1916 Prediction

- **What creates gravitational waves?**
  - Any asymmetrically accelerating mass sends ripples through spacetime.
  - Strongest sources: binary systems spiralling together.
  - As energy is radiated away, the orbit tightens and accelerates.
  - First predicted by Einstein in 1916 from the field equations
  - Discovered by Hulse & Taylor (1974) at Arecibo: two neutron stars orbiting each other every ~7.75 hours.
  - LIGO detected the wave itself directly in 2015 (Nobel 2017).



$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

[3.02, 3.03]

What exactly IS a gravitational wave?

- Not a wave of matter or electromagnetic energy, it is a wave of spacetime geometry itself.

- As it passes, space physically stretches in one direction and squeezes in the perpendicular one, alternating at the wave frequency

- No medium required, travels through empty space.

- LIGO arms: 4 km long, stretched/squeezed by less than 1/1000th the width of a proton.

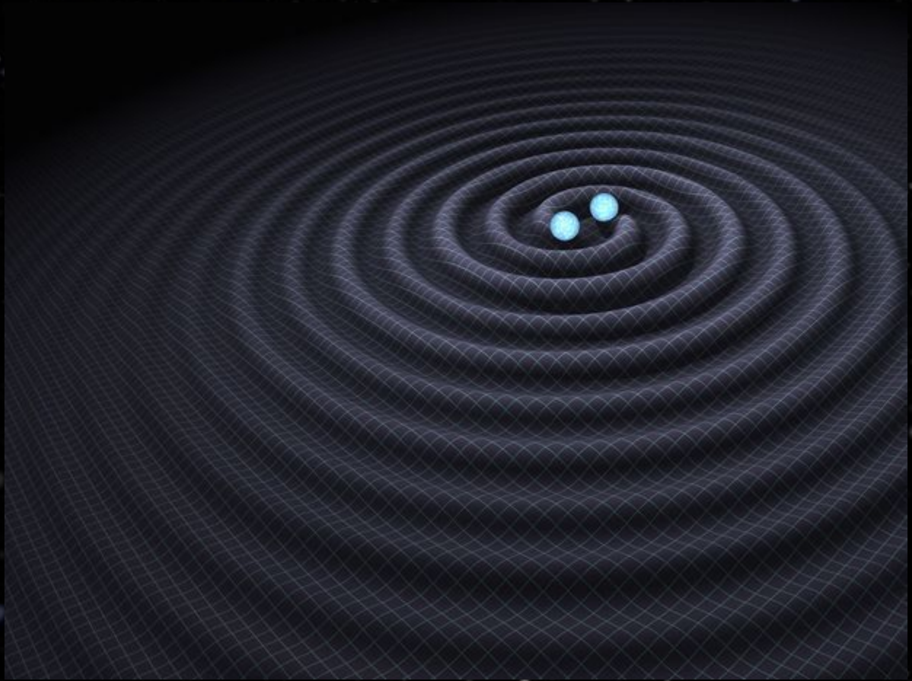


Figure 3.4: Caltech/MIT/LIGO Lab. (2016). ligo20160211f. Retrieved from https://www.ligo.caltech.edu/image/ligo20160211f

What Does Curved Spacetime Actually Look Like?

- **The Schwarzschild solution (1916)**
  - Schwarzschild solved Einstein's equations for a single spherical mass.
  - First exact solution, derived within weeks of GR being published.
  - Describes the geometry outside any spherical mass: stars, planets.
- **The Flamm Paraboloid: the geometry made visible**
  - Ludwig Flamm (1916) embedded the Schwarzschild geometry as a paraboloid in 3D the shape of the 'rubber sheet' analogy.
  - Our simulation draws this surface live.

Figure 3.5: Karl Schwarzschild [Photograph]. (c. 1910). Wikimedia Commons. Retrieved from https://en.wikipedia.org/wiki/Karl_Schwarzschild

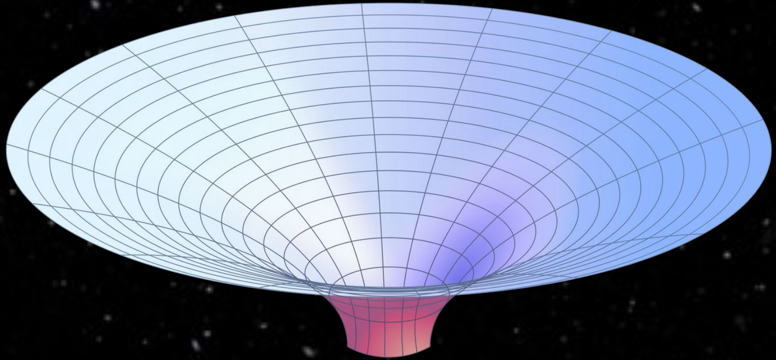
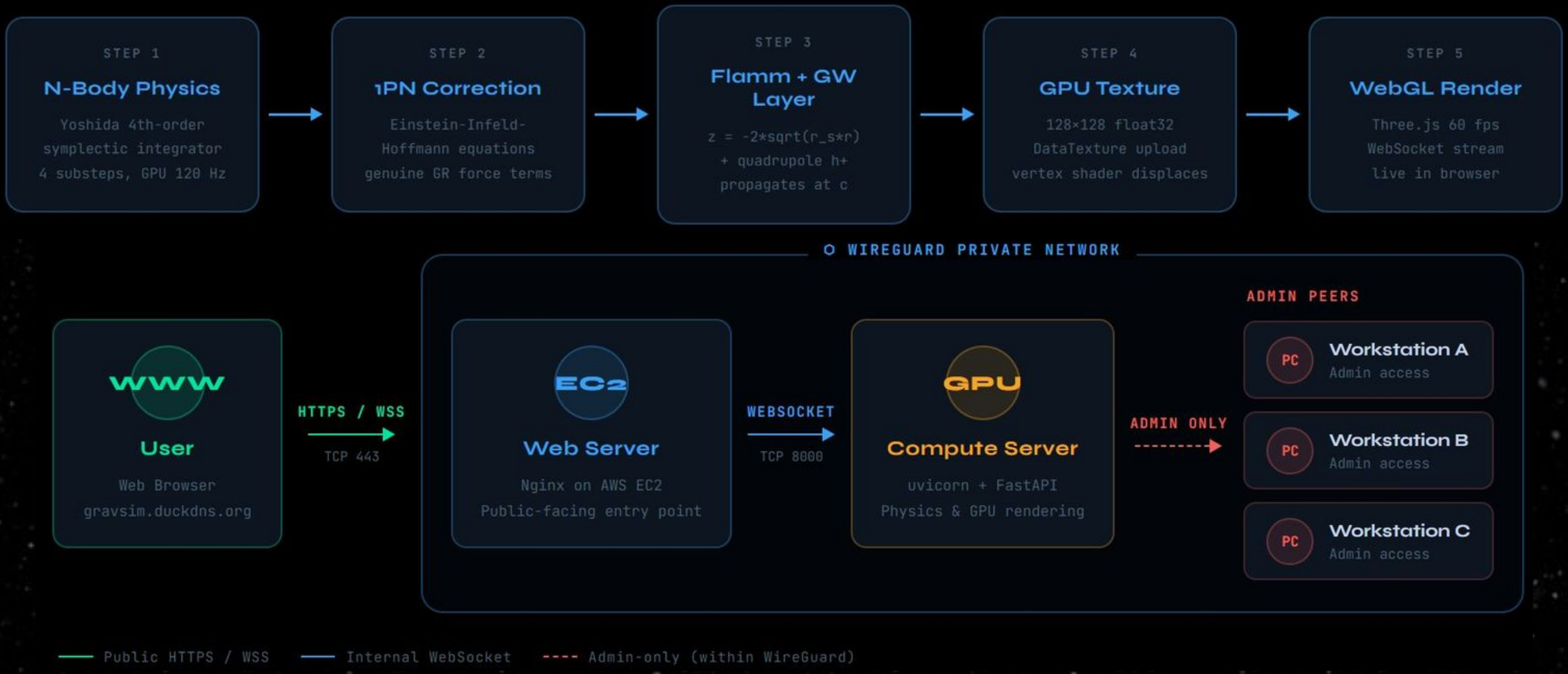


Figure 3.6: Flamm's paraboloid [Digital illustration]. Wikimedia Commons. Retrieved from https://en.wikipedia.org/wiki/Schwarzschild_metric#Flamm's_paraboloid

Simulating Einstein's Universe in Real Time



From Theory to Observation

How well does our simulation compare to reality?

Background: Orbital Metrics

Orbital Period

The time it takes an astronomical object to complete on full revolution around another

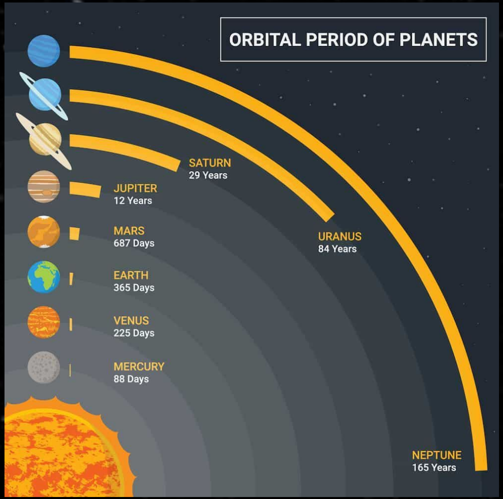


Figure 4.01. Orbital period of Planets. Retrieved from https://starlust.org/orbital-parameters-glossary/

Eccentricity

- e = 0: Perfect Circle
- 0 < e < 1: ellipse
- e = 1: Parabola

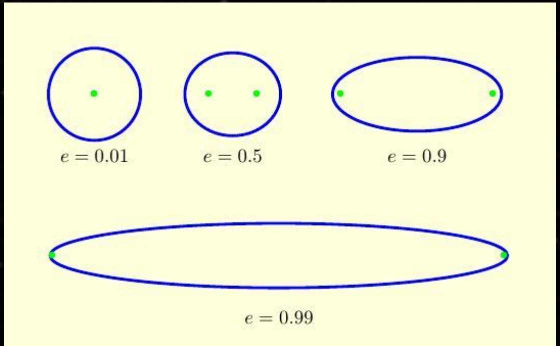


Figure 4.02 Ellipses. Retrieved from https://web.ma.utexas.edu/users/m408m/LM10-5-3.html

Hypothesis

Our simulation should behave similar to reality, with some minor differences due to our initial conditions

Initial Conditions:

- Planets start collinear (in a line)
- Planets placed at their average AU distance from the Sun

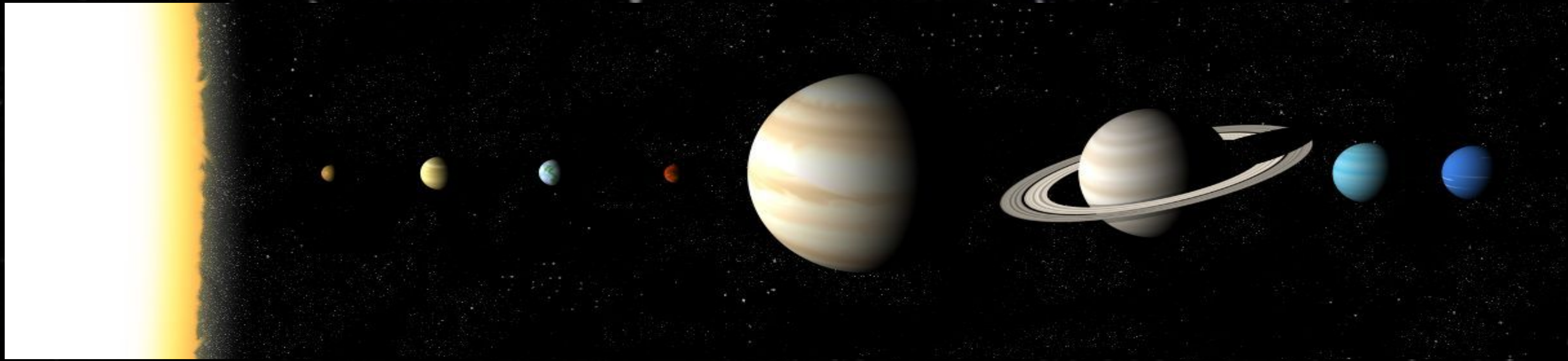


Figure 4.03. Planets. Retrieved from https://www.wtamu.edu/~cbaird/sq/2013/08/28/when-do-the-planets-in-our-solar-system-all-line-up/

Methodology

How I Measured Orbital Period

Track the orbital angle θ around the barycentre

When accumulated angles reaches 2π

- one orbit has been completed
- Record the time

How I Measured Eccentricity

Eccentricity Vector Method

$$e = \frac{\vec{v} \times \vec{h}}{\mu} - \frac{\vec{r}}{|\vec{r}|}$$

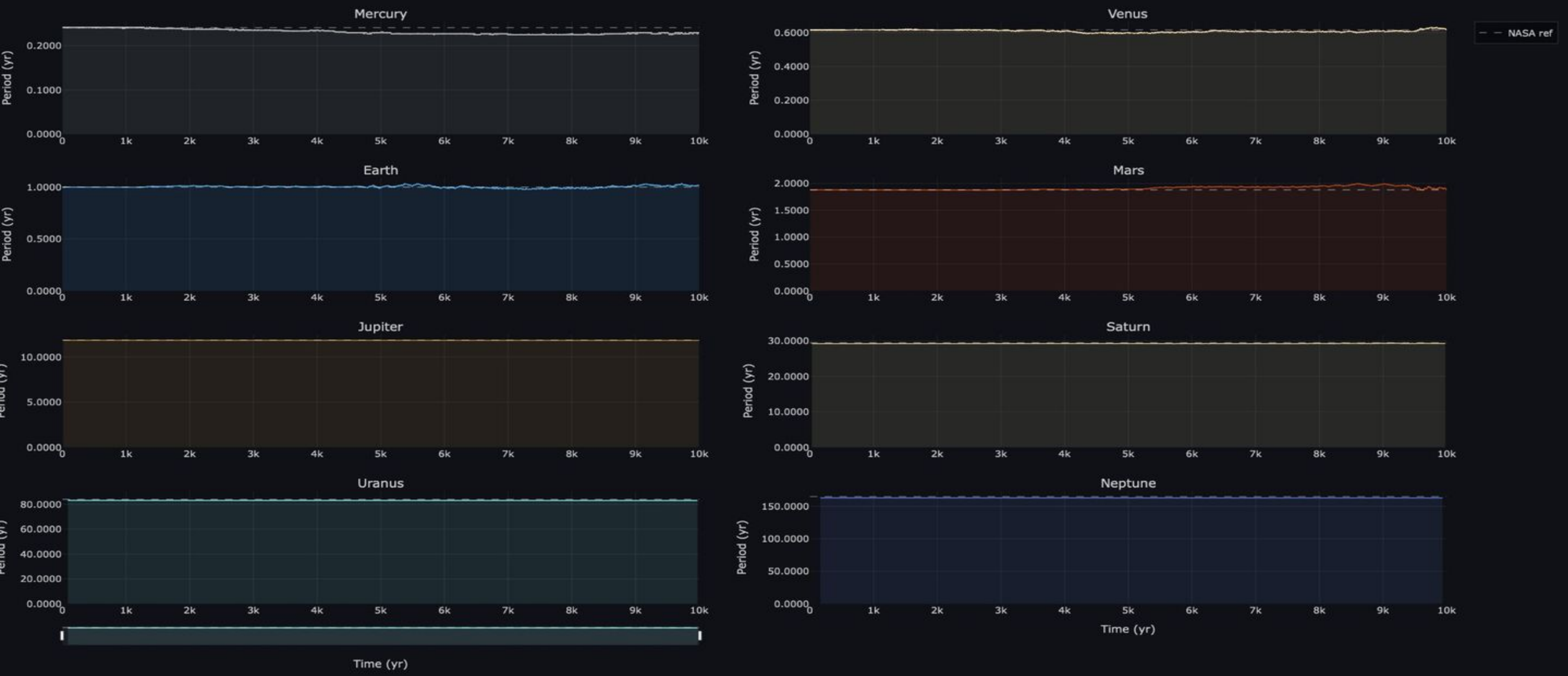
$\mathcal{V}$ = the velocity vector
h = specific angular momentum vector (r x v)
μ = the standard gravitational parameter}}

|e| = the orbits eccentricity

Check after each step.
Average over the entire orbit

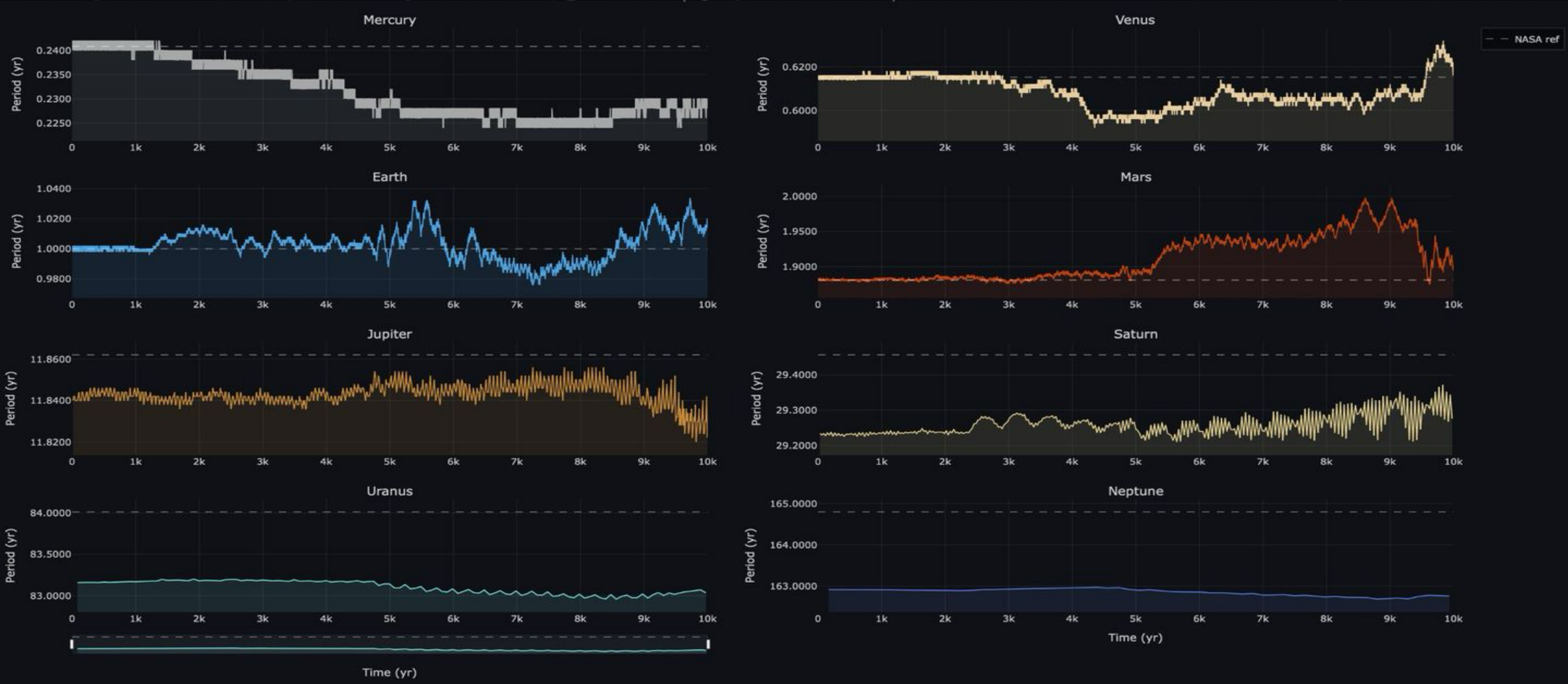
Refs: [4.02]

Solar System Simulation - Orbital Period - 10,000 Years



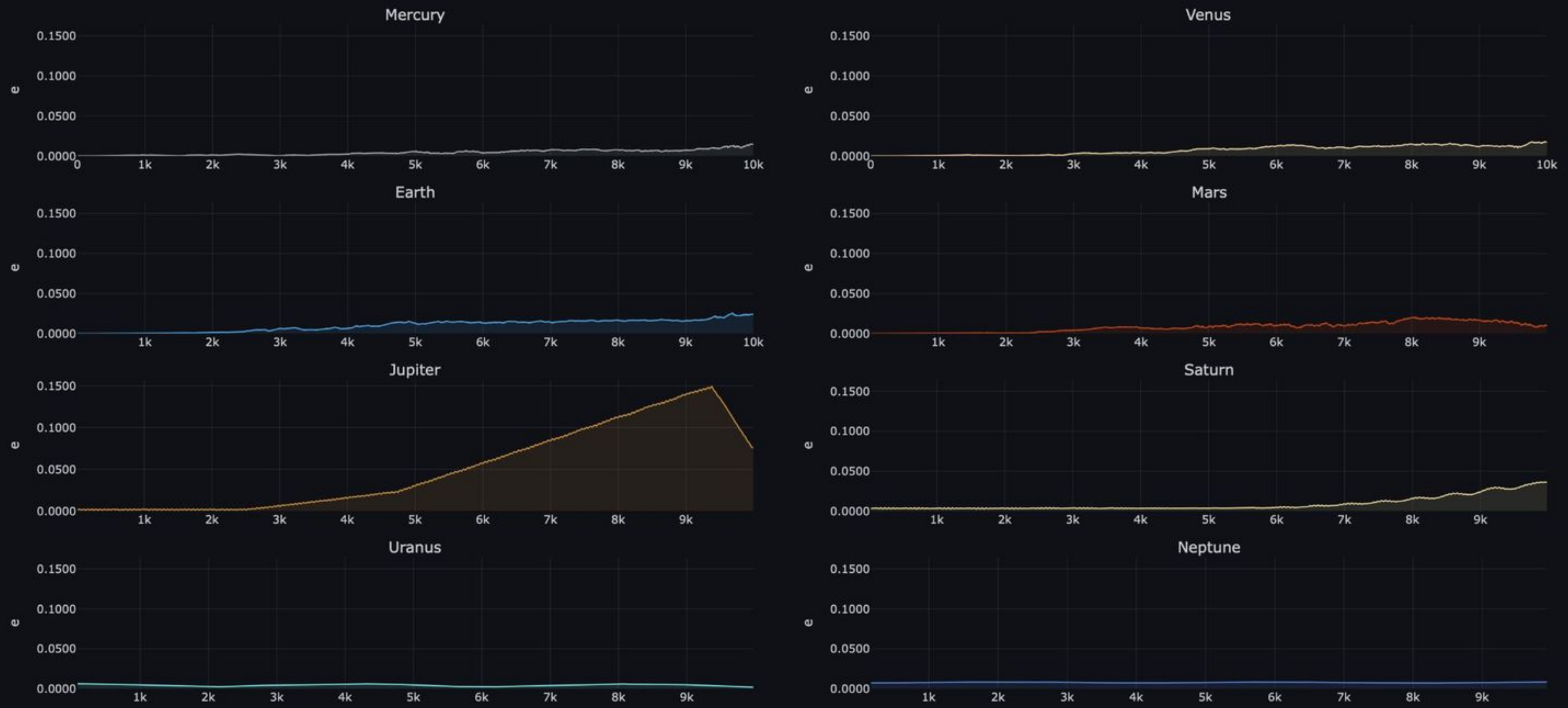
Refs: [4.01]]

Solar System Simulation - Orbital Period - 10,000 Years

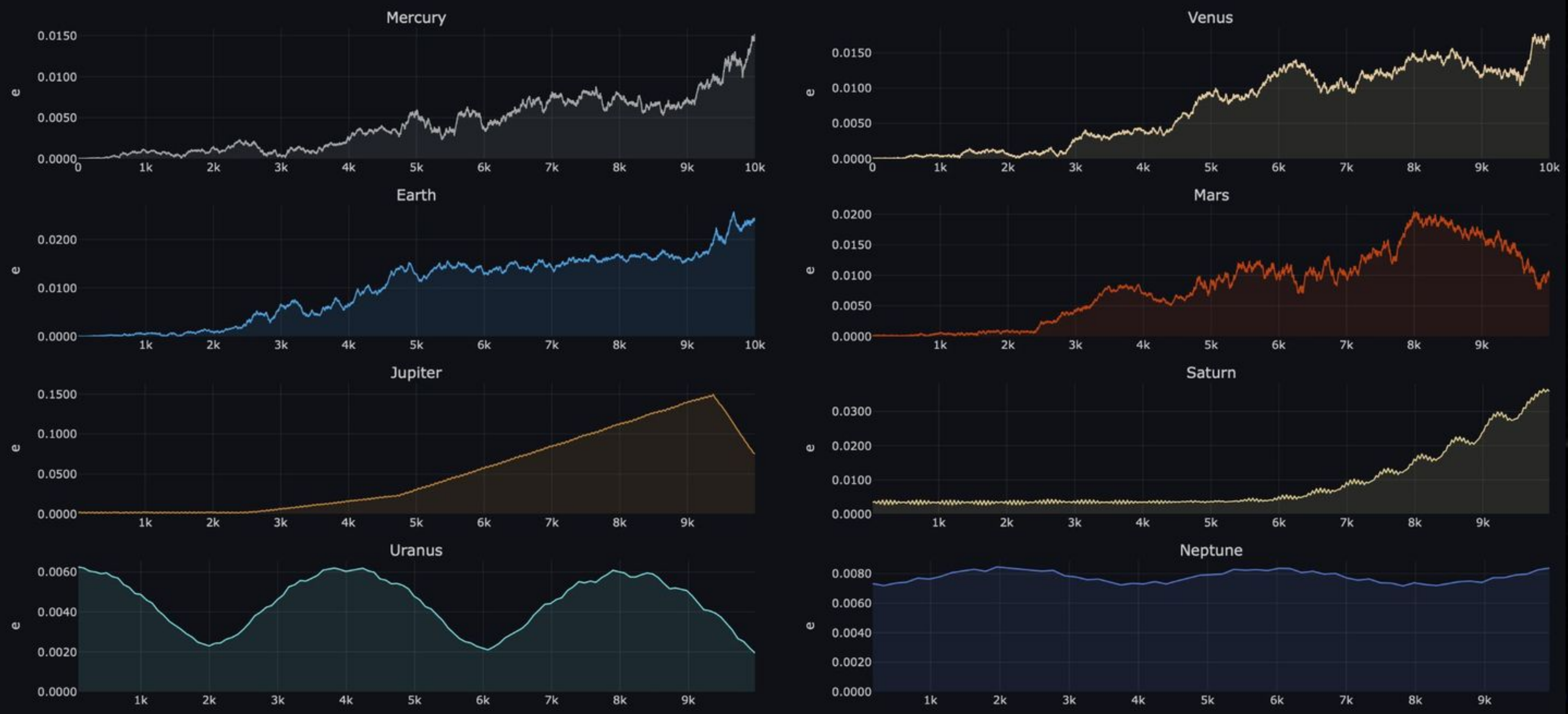


Refs: [4.01]]

Solar System Simulation - Eccentricity - 10,000 Years



Solar System Simulation - Eccentricity - 10,000 Years



Solar System Simulation - Eccentricity - 10,000 Years

Refs: [4.01]]

Initial Conditions Determine Everything

Our Simulation

Clean, Idealized start:

- All planets aligned (collinear)
- Circular orbits assumed
- No collision history

➔ Leads to instability
➔ Eccentricities grow overtime

Real Solar System

4.6 billion years of evolution:

- Formed from rotating disk of gas
- Late heavy bombardment era
- Planetary collisions and ejections
- Gradual orbital settling

➔ Current orbits are survivors
➔ Chaotic past → stable present

Questions?